Published by Institute of Physics Publishing for SISSA

Received: *September 3, 2007*
Revised: *November 8, 2007*
Accepted: *December 18, 2007*
Published: *January 3, 2008*

Nonperturbative ghost dynamics in the maximal abelian gauge

M.A.L. Capri,[a] **D. Dudal,**[b] **J.A. Gracey,**[c] **S.P. Sorella**[a] **and H. Verschelde**[b]

[a] *Departamento de Física Teórica,*
Instituto de Física, UERJ, Universidade do Estado do Rio de Janeiro,
Rua São Francisco Xavier 524, 20550-013 Maracanã, Rio de Janeiro, Brasil
[b] *Department of Mathematical Physics and Astronomy, Ghent University,*
Krijgslaan 281-S9, B-9000 Gent, Belgium
[c] *Theoretical Physics Division, Department of Mathematical Sciences,*
University of Liverpool,
P.O. Box 147, Liverpool, L69 3BX, U.K.
E-mail: marcio@dft.if.uerj.br, david.dudal@ugent.be, gracey@liv.ac.uk, sorella@uerj.br, henri.verschelde@ugent.be

Abstract: We construct the effective potential for the ghost condensate $\langle f^{abi}\overline{c}^a c^b\rangle$ in the maximal Abelian gauge. This condensate is an order parameter for a global continuous symmetry, which is spontaneously broken since a nonvanishing value of $\langle f^{abi}\overline{c}^a c^b\rangle$ lowers the vacuum energy. The associated Goldstone mode turns out to be unphysical.

Keywords: Nonperturbative Effects, BRST Quantization, Renormalization Group, Spontaneous Symmetry Breaking.

JHEP01(2008)006

Contents

1. Introduction

Perturbatively, Faddeev-Popov ghosts are well understood. In textbooks [1], these anticommuting scalar fields are usually introduced as a tool to lift the Faddeev-Popov determinant into the action. This determinant is the Jacobian arising from the gauge fixing condition. After this, a consistent perturbative expansion of the path integral can be carried out.

However, ghosts are much more than a "mathematical trick". In a sense, they naturally arise when a gauge is fixed in a Lorentz covariant manner. Once the gauge is fixed, the (local) gauge freedom, generated by $\delta_\omega A^a_\mu = D^{ab}_\mu \omega^b$, is of course lost. However, we recover a BRST symmetry of the complete action $S_{\rm YM} + S_{\rm gf}$, such that the gauge fixing part of the action is BRST exact, i.e. $S_{\rm gf} = sS'_{\rm gf}$, where s is the nilpotent BRST operator.

A more general class of gauge fixings than those which can be obtained through the Faddeev-Popov method, can be introduced by making direct use of the BRST symmetry: one adds a BRST-exact expression $sS'_{\rm gf}$ to the classical Yang-Mills action in order to break the gauge invariance.

Clearly, the ghosts play a crucial role in this construction. The BRST symmetry can be used to prove e.g. the renormalizability of gauge theories, the unitarity of the S-matrix, the gauge parameter independence of gauge invariant correlation functions, etc.

Ghosts also have a clear physical meaning at the perturbative level, although *a clear unphysical meaning* would perhaps be a better choice of words. Indeed, asymptotically, their degrees of freedom cancel out with the scalar and longitudinal gauge boson polarizations from a suitably defined physical subspace, leaving two physical transverse polarizations, as desired [2].

Perhaps less commonly known is that ghosts can also be used to discuss quantum properties of anomalies. For example, the ghost polynomials $\mathrm{Tr} c^{2k+1}$ can be used to discuss nonrenormalization properties of e.g. the gauge anomaly (Adler-Bardeen theorem). We refer to [3] for relevant details and original literature.

This very short summary should have sufficiently outlined the relevance of ghosts. The reader will have noticed that all the previous results are strictly speaking at the perturbative level. A natural question is what the role of the ghosts might become when going beyond the perturbative level? Of course, since ghosts arise only *after* gauge fixing, one might question the issue of gauge invariance. But as we have to add them to our action at the quantum level, we cannot disregard the possibility that ghosts might be important for the infrared dynamics of gauge theories, signalling certain nonperturbative effects in at least a particular gauge. Let us quote two cases in which ghosts are relevant at the nonperturbative level. Firstly, when gauge copies in the Landau gauge are taken into account by implementing the restriction of the domain of integration in the functional integral to the so called Gribov region Ω [4], the ghost propagator is found to behave like $\frac{1}{q^4}$ for $q^2 \sim 0$. At the same time, an infrared suppressed gluon propagator is found. This leads to an infrared fixed point of the nonperturbatively defined strong coupling constant [5], and induces a violation of positivity of the gluon propagator, indicating that the gluon cannot be a stable asymptotic physical particle [6]. Secondly, Kugo and Ojima constructed an algebraic criterion for confinement (an inherent nonperturbative infrared phenomenon), which is fulfilled in the Landau gauge when the ghost propagator is sufficiently singular [7, 8]. The infrared enhancement of the ghost propagator and the infrared suppression of the gluon propagator have received confirmation from lattice simulations [9] as well as from the study of the Schwinger-Dyson equations [10 – 12].

In this paper, we will elaborate on another intriguing possibility: the formation of a nonperturbative ghost condensate. This problem was tackled first in [13, 14] in case of the maximal Abelian gauge (MAG). The MAG Yang-Mills action is given by[1]

$$S = S_{\rm YM} + S_{\rm MAG} + S_{\rm diag} \,, \tag{1.1}$$

whereby

$$S_{\rm YM} = -\frac{1}{4}\int d^4x \left(F^a_{\mu\nu}F^{a\mu\nu} + F^i_{\mu\nu}F^{i\mu\nu}\right) , \tag{1.2}$$

$$\begin{aligned} S_{\rm MAG} &= s\overline{s}\int d^4x \left(\frac{1}{2}A^a_\mu A^{a\mu} - \frac{\alpha}{2}c^a\overline{c}^a\right) \qquad (1.3)\\ &= \int d^4x \left[b^a\left(D^{ab}_\mu A^{b\mu} + \frac{\alpha}{2}b^a\right) + \overline{c}^a D^{ab}_\mu D^{bc\mu}c^c + g f^{abi}\overline{c}^a (D^{bc}_\mu A^{c\mu})c^i \right.\\ &\qquad + g f^{bcd}\overline{c}^a D^{ab}_\mu (A^{c\mu}c^d) - \alpha g f^{abi} b^a \overline{c}^b c^i - g^2 f^{abi} f^{cdi}\overline{c}^a c^d A^b_\mu A^{c\mu} - \frac{\alpha}{2} g f^{abc} b^a \overline{c}^b c^c\\ &\qquad \left. - \frac{\alpha}{4} g^2 f^{abi} f^{cdi}\overline{c}^a\overline{c}^b c^c c^d - \frac{\alpha}{4} g^2 f^{abc} f^{adi}\overline{c}^b\overline{c}^c c^d c^i - \frac{\alpha}{8} g^2 f^{abc} f^{ade}\overline{c}^b\overline{c}^c c^d c^e\right] , \end{aligned}$$

$$S_{\rm diag} = s\int d^4x\, \overline{c}^i \partial^\mu A^i_\mu = \int d^4x \left[b^i \partial^\mu A^i_\mu + \overline{c}^i \partial^\mu (\partial_\mu c^i + g f^{abi} A^a_\mu c^b)\right] , \tag{1.4}$$

with

$$D^{ab}_\mu = \delta^{ab}\partial_\mu - g f^{abi} A^i_\mu \tag{1.5}$$

[1] Indices like $a, b, \ldots$ refer to the off-diagonal sector, while $i, j, \ldots$ to the diagonal one.

the $\mathrm{U}(1)^{N-1}$ covariant derivative and

$$F^a_{\mu\nu} = D^{ab}_\mu A^b_\nu - D^{ab}_\nu A^b_\mu + g f^{abc} A^b_\mu A^c_\nu \, , F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g f^{abi} A^a_\mu A^b_\nu \tag{1.6}$$

the field strength. The nilpotent (anti)-BRST transformations of the fields read as follows

$$\begin{aligned} sA^a_\mu &= -(D^{ab}_\mu c^b + g f^{abc} A^b_\mu c^c + g f^{abi} A^b_\mu c^i) \, , & sA^i_\mu &= -(\partial_\mu c^i + g f^{abi} A^a_\mu c^b) \, , \\ sc^a &= g f^{abi} c^b c^i + \frac{g}{2} f^{abc} c^b c^c \, , & sc^i &= \frac{g}{2} f^{abi} c^a c^b \, , \\ s\overline{c}^a &= b^a \, , sb^a = 0 \, , & s\overline{c}^i &= b^i \, , sb^i = 0 \, , \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} \overline{s}A^a_\mu &= -(D^{ab}_\mu \overline{c}^b + g f^{abc} A^b_\mu \overline{c}^c + g f^{abi} A^b_\mu \overline{c}^i) \, , & \overline{s}A^i_\mu &= -(\partial_\mu \overline{c}^i + g f^{abi} A^a_\mu \overline{c}^b) \, , \\ \overline{sc}^a &= g f^{abi} \overline{c}^b \overline{c}^i + \frac{g}{2} f^{abc} \overline{c}^b \overline{c}^c \, , & \overline{sc}^i &= \frac{g}{2} f^{abi} \overline{c}^a \overline{c}^b \, , \\ \overline{s}c^a &= -b^a + g f^{abc} c^b \overline{c}^c + g f^{abi} c^b \overline{c}^i + g f^{abi} \overline{c}^b c^i \, , \\ \overline{s}c^i &= -b^i + g f^{abi} c^a \overline{c}^b \, , & \overline{s}b^a &= -g f^{abc} b^b \overline{c}^c - g f^{abi} b^b \overline{c}^i + g f^{abi} \overline{c}^b b^i \;\; \overline{s}b^i = -g f^{abi} b^a \overline{c}^b \, . \end{aligned} \tag{1.8}$$

We also recall the Jacobi identity, in decomposed form [21]

$$f^{abi} f^{bjc} + f^{abj} f^{bci} = 0 \, , \qquad f^{abc} f^{bdi} + f^{abd} f^{bic} + f^{abi} f^{bcd} = 0 \, . \tag{1.9}$$

Strictly speaking, the MAG is defined by choosing that gauge configuration that corresponds to the (absolute) minimum of the functional

$$\mathcal{R}_{\mathrm{MAG}}[A] = \int d^4x (A^a_\mu)^2 \tag{1.10}$$

under gauge variations. Restricting to infinitesimal gauge variations, it reduces to the $\mathrm{U}(1)^{N-1}$ covariant constraint

$$D^{ab}_\mu A^{b\mu} = 0 \, . \tag{1.11}$$

The residual Abelian gauge freedom is fixed by a Landau like condition, (1.4). We notice that this gauge fixing for the diagonal part does not exhibit the anti-BRST symmetry. We have only introduced the anti-BRST transformation $\overline{s}$ as a tool to write down a condensed form of the off-diagonal gauge fixing, i.e. the MAG (1.3). As a consequence, only the BRST symmetry and its associated Slavnov-Taylor identity will be used for the renormalization analysis.

The MAG has received much interest, as it might be relevant for the dual superconductivity picture of confinement [15]. To make the MAG well-defined at the perturbative level, one must introduce a regulating gauge parameter α and add a 4-point ghost interaction proportional to α to the action [16, 17]. The condition (1.11) is retrieved in the formal limit $\alpha \to 0$. As is well known from other models, in the presence of an attractive four fermion interaction, the formation of a fermion condensate can become energetically favoured. In our case, the analog phenomenon would be the formation of a ghost condensate. This was

originally discussed in [13, 14] by a decomposition of the 4-point interaction by means of an auxiliary field σ. A one loop effective potential was constructed, and a nonvanishing condensate $\langle\sigma\rangle$, proportional to the ghost condensate, was found. However, as explained in [18], this gives rise to problems with the renormalization group (RG) beyond the one loop level. Next to this, it is also of no use in the Landau gauge as there is no 4-point interaction present in that case. In this article, we shall invoke the LCO formalism, originated in [19] by one of us, which allows a RG consistent discussion of Local Composite Operators. We shall also make use of results obtained in a series of papers about the usage of the LCO formalism in gauge theories [20, 21]. The ghost condensate was used in [13, 14] to generate a dynamical off-diagonal gluon mass. As a consequence, the off-diagonal gluons should decouple from the infrared dynamics, hinting that an Abelian theory could be used to eventually obtain confinement. It was however realized in [18] that the effective off-diagonal gluon mass was tachyonic, and therefore certainly not suitable to explain the so-called Abelian dominance [22].

Nevertheless, this does not mean that the ghost condensate is meaningless. The fact that it gives rise to a tachyonic effective gluon mass points out that other condensates might emerge. Indeed, it was discussed in [21, 23] that a mixed gluon-ghost operator condenses and gives rise to a real valued effective off-diagonal mass, a result in qualitative accordance with available lattice simulations in the MAG [24, 25].

In the Landau gauge, we already presented a combined study of the gauge condensate $\langle A^2\rangle$ together with the ghost condensate [26]. The one loop net result is that the tachyonic contribution of the ghost condensate induces a splitting between the diagonal and off-diagonal mass, leaving a larger value for the off-diagonal one. This can be seen as evidence for some kind of Abelian dominance in the Landau gauge [27].

A nontrivial condensation is frequently intimately entangled with the spontaneous breaking of some global symmetry. In fact, as discussed in [13, 14, 20], the ghost condensation breaks a global invariance present in the maximal Abelian gauge, generated by

$$\delta\overline{c}^a = c^a \ , \quad \delta b^a = \frac{g}{2} f^{abc} c^b c^c + g f^{abi} c^b c^i \ , \quad \delta(\text{rest}) = 0 \ . \tag{1.12}$$

As a version of this symmetry is also present in the Curci-Ferrari , see [28], and Landau gauges [29], it might be expected that a ghost condensation could occur in these gauges too. This point was discussed in [30 – 32]. By choosing another diagonal gauge fixing in the case of the MAG [29], it is possible to find an even larger symmetry content, next to δ. More precisely, it is possible to find a complete $\mathrm{SL}(2,\mathbb{R})$ invariance, generated by the ghost number symmetry generator δ_c, δ and an analogue of δ with an exchange of ghost-antighost fields. In the Landau gauge, the $\mathrm{SL}(2,\mathbb{R})$ rotations connect different channels of ghost condensation. Next to the operator $f^{abc}\overline{c}^b c^c$ with vanishing ghost number, also the ghost charged operators $f^{abc}\overline{c}^b\overline{c}^c$ and $f^{abc}c^b c^c$ can condense in principle. However, the corresponding vacua are equivalent [32], and for simplicity we will restrict ourselves in this paper to the uncharged channel $f^{abi}\overline{c}^a c^b$ instead of the charged ones ($f^{abi}c^a c^b / f^{abi}\overline{c}^a\overline{c}^b$). In [20], both channels were discussed simultaneously in the Landau gauge.

Let us finally mention that, precursored by the theoretical results, also lattice studies have been made in the Landau as well as in the maximal Abelian gauge, giving sup-

JHEP01(2008)006

port to the existence of a nonvanishing ghost condensate and dynamically broken symmetry [33, 34].

This article is organized as follows: in the second section, we set up the action and prove the renormalizability in the presence of the composite ghost operator $f^{abi}\overline{c}^a c^b$ and of the dimension two mass operator $(\frac{1}{2}A^a_\mu A^a_\mu + \alpha\overline{c}^a c^a)$ using the Ward identities of the MAG. In the third section, we derive the necessary RG functions and discuss the one loop ghost condensation for the gauge group SU(2). Section 4 summarizes some consequences of a nontrivial ghost condensate. We also prove that the Goldstone boson corresponding to the spontaneously broken δ-symmetry decouples from the physical spectrum. The last section contains our conclusions.

2. The action: algebraic analysis

The complete action we start with, reads

$$\Sigma = S_{\rm YM} + S_{\rm MAG} + S_{\rm diag} + S_{\rm LCO} + S_{\rm ext} \ , \tag{2.1}$$

where,

$$\begin{aligned} S_{\rm LCO} &= s\int d^4x \left[\lambda\left(\frac{1}{2}A^a_\mu A^{a\mu} + \alpha\overline{c}^a c^a\right) + \frac{\zeta}{2}\lambda J + g f^{abi}\omega^i\overline{c}^a c^b + \frac{\chi}{2}\omega^i\vartheta^i\right] \\ &= \int d^4x \left[J\left(\frac{1}{2}A^a_\mu A^{a\mu} + \alpha\overline{c}^a c^a\right) + \frac{\zeta}{2}J^2 - \alpha\lambda b^a c^a + \lambda A^{a\mu}D^{ab}_\mu c^b + \alpha\lambda g f^{abi}\overline{c}^a c^b c^i \right. \\ &\quad + \alpha\lambda\frac{g}{2}f^{abc}\overline{c}^a c^b c^c + g f^{abi}\vartheta^i\overline{c}^a c^b - g f^{abi}\omega^i b^a c^b + g^2 f^{abi}f^{bcj}\omega^i\overline{c}^a c^c c^j \\ &\quad \left. + \frac{g^2}{2}f^{abi}f^{bcd}\omega^i\overline{c}^a c^c c^d + \frac{\chi}{2}\vartheta^i\vartheta^i\right] , \end{aligned} \tag{2.2}$$

$$\begin{aligned} S_{\rm ext} &= \int d^4x \left[-\Omega^{a\mu}\Big(D^{ab}_\mu c^b + g f^{abc}A^b_\mu c^c + g f^{abi}A^b_\mu c^i\Big) - \Omega^{i\mu}\Big(\partial_\mu c^i + g f^{abi}A^a_\mu c^b\Big)\right. \\ &\quad \left. + L^a\left(g f^{abi}c^b c^i + \frac{g}{2}f^{abc}c^b c^c\right) + \frac{g}{2}f^{abi}L^i c^a c^b\right] . \end{aligned} \tag{2.3}$$

The external sources $\Omega^{a,i}_\mu$ and $L^{a,i}$ are needed to define the composite operators entering the nonlinear BRST transformations of the field $A^{a,i}_\mu$ and $c^{a,i}$, respectively. These sources are invariant under the action of the BRST operator, i.e.,

$$s\Omega^a_\mu = s\Omega^i_\mu = 0 \ , \qquad sL^a = sL^i = 0. \tag{2.4}$$

The two pairs of sources (J,λ) and (ϑ^i,ω^i) are needed in order to define the composite operators $\left(A^a_\mu A^{a\mu} + \alpha\overline{c}^a c^c\right)$ and $g f^{abi}\overline{c}^a c^b$ and their BRST variations. These sources form BRST doublets, according to

$$s\lambda = J, \qquad sJ = 0 \ , \qquad s\omega^i = \vartheta^i, \qquad s\vartheta^i = 0. \tag{2.5}$$

The purpose of the pure source terms $\frac{\chi}{2}\vartheta^i\vartheta^i$ and $\frac{\zeta}{2}J^2$ shall be made clear in the next section. The mass dimension and the ghost number of the fields and sources have been listed in table 1. The complete action (2.1) obeys the following set of Ward identities:

	$A^{a,i}_\mu$	$c^{a,i}$	$\overline{c}^{a,i}$	$b^{a,i}$	λ	J	$\Omega^{a,i}_\mu$	$L^{a,i}$	ω^i	ϑ^i
dimension	1	0	2	2	2	2	3	4	2	2
ghost number	0	1	-1	0	-1	0	-1	-2	-1	0

Table 1: Quantum numbers of the field and sources

- The Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) = \int d^4x \left(\frac{\delta\Sigma}{\delta\Omega^{a\mu}}\frac{\delta\Sigma}{\delta A^a_\mu} + \frac{\delta\Sigma}{\delta\Omega^{i\mu}}\frac{\delta\Sigma}{\delta A^i_\mu} + \frac{\delta\Sigma}{\delta L^a}\frac{\delta\Sigma}{\delta c^a} + \frac{\delta\Sigma}{\delta L^i}\frac{\delta\Sigma}{\delta c^i} + b^a\frac{\delta\Sigma}{\delta\overline{c}^a} + b^i\frac{\delta\Sigma}{\delta\overline{c}^i} + \vartheta^i\frac{\delta\Sigma}{\delta\omega^i} + J\frac{\delta\Sigma}{\delta\lambda}\right) = 0 \,, \tag{2.6}$$

- The diagonal ghost equation

$$\begin{aligned} \mathcal{G}^i(\Sigma) &= \Delta^i_{\text{class}} \,, \\ \mathcal{G}^i &= \frac{\delta}{\delta c^i} + g f^{abi}\overline{c}^a\frac{\delta}{\delta b^b} \,, \\ \Delta^i_{\text{class}} &= -\partial^2\overline{c}^i + g f^{abi}\Omega^{a\mu}A^b_\mu - \partial_\mu\Omega^{i\mu} - g f^{abi}L^a c^b \,. \end{aligned} \tag{2.7}$$

- The diagonal gauge-fixing condition

$$\frac{\delta\Sigma}{\delta b^i} = \partial_\mu A^{i\mu} \,. \tag{2.8}$$

- The anti-ghost equation

$$\overline{\mathcal{G}}^i(\Sigma) = \frac{\delta\Sigma}{\delta\overline{c}^i} + \partial^\mu\frac{\delta\Sigma}{\delta\Omega^{i\mu}} = 0 \,. \tag{2.9}$$

- The diagonal $\mathrm{U}(1)^{N-1}$ Ward identity

$$\begin{aligned} \mathcal{W}^i(\Sigma) &= -\partial^2 b^i \,, \\ \mathcal{W}^i &= \partial_\mu\frac{\delta}{\delta A^i_\mu} + g f^{abi}\left(A^a_\mu\frac{\delta}{\delta A^b_\mu} + c^a\frac{\delta}{\delta c^b} + b^a\frac{\delta}{\delta b^b} + \overline{c}^a\frac{\delta}{\delta\overline{c}^b} + \Omega^{a\mu}\frac{\delta}{\delta\Omega^{b\mu}} + L^a\frac{\delta}{\delta L^b}\right) . \end{aligned} \tag{2.10}$$

- The integrated λ-equation

$$\mathcal{U}(\Sigma) = \int d^4x \left(\frac{\delta\Sigma}{\delta\lambda} + c^a\frac{\delta\Sigma}{\delta b^a} - 2\omega^i\frac{\delta\Sigma}{\delta L^i}\right) = 0 \,. \tag{2.11}$$

- The $\mathrm{SL}(2,\mathbb{R})$ Ward identity

$$D(\Sigma) = \int d^4x \left(c^a\frac{\delta\Sigma}{\delta\overline{c}^a} + \frac{\delta\Sigma}{\delta L^a}\frac{\delta\Sigma}{\delta b^a} - 2\vartheta^i\frac{\delta\Sigma}{\delta L^i}\right) = 0 \,. \tag{2.12}$$

We notice that the terms $\Delta^i_{\rm class}$, in (2.7), and $-\partial^2 b^i$, in (2.10), are linear in the quantum fields, thus defining classical breakings.

We are now ready to write down the most general counterterm, $\Sigma_{\rm CT}$, which is compatible with the previous Ward identities and which can be freely added to the original action. Requiring that the perturbed action, $\Sigma + \eta\Sigma_{\rm CT}$, obeys the same Ward identities as Σ to the first order in the expansion parameter η, one gets the following conditions

$$\mathcal{B}_\Sigma \Sigma_{\rm CT} = 0 \,, \frac{\delta \Sigma_{\rm CT}}{\delta b^i} = 0 \,, \mathcal{G}^i \Sigma_{\rm CT} = 0 \,, \mathcal{W}^i \Sigma_{\rm CT} = 0 \,, \overline{\mathcal{G}}^i \Sigma_{\rm CT} = 0, \mathcal{U} \Sigma_{\rm CT} = 0 \,, D_\Sigma \Sigma_{\rm CT} = 0 \,, \tag{2.13}$$

where $\mathcal{B}_\Sigma$ is the nilpotent, $\mathcal{B}^2_\Sigma = 0$, linearized Slavnov-Taylor operator, given by

$$\begin{aligned}\mathcal{B}_\Sigma = \int d^4x \Bigg(&\frac{\delta \Sigma}{\delta \Omega^{a\mu}} \frac{\delta}{\delta A^a_\mu} + \frac{\delta \Sigma}{\delta A^a_\mu} \frac{\delta}{\delta \Omega^{a\mu}} + \frac{\delta \Sigma}{\delta \Omega^{i\mu}} \frac{\delta}{\delta A^i_\mu} + \frac{\delta \Sigma}{\delta A^i_\mu} \frac{\delta}{\delta \Omega^{i\mu}} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta}{\delta c^a} + \\ &+ \frac{\delta \Sigma}{\delta c^a} \frac{\delta}{\delta L^a} + \frac{\delta \Sigma}{\delta L^i} \frac{\delta}{\delta c^i} + \frac{\delta \Sigma}{\delta c^i} \frac{\delta}{\delta L^i} + b^a \frac{\delta}{\delta \overline{c}^a} + b^i \frac{\delta}{\delta \overline{c}^i} + \vartheta^i \frac{\delta}{\delta \omega^i} + J \frac{\delta}{\delta \lambda} \Bigg),\end{aligned} \tag{2.14}$$

and the operator D_Σ, in (2.13), is given by:

$$D_\Sigma = \int d^4x \left(c^a \frac{\delta}{\delta \overline{c}^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta}{\delta b^a} + \frac{\delta \Sigma}{\delta b^a} \frac{\delta}{\delta L^a} - 2\vartheta^i \frac{\delta}{\delta L^i} \right) . \tag{2.15}$$

The most general local counterterm can be written as

$$\Sigma_{\rm CT} = a_0 \, S_{\rm YM} + \mathcal{B}_\Sigma \Delta^{(-1)} \,, \tag{2.16}$$

where $\Delta^{(-1)}$ is an integrated local polynomial of ghost number -1 and dimension 4, given by:

$$\begin{aligned}\Delta^{(-1)} = \int d^4x \Big[&a_1 \, \Omega^{i\mu} A^i_\mu + a_2 \, (\partial^\mu \overline{c}^i) A^i_\mu + a_3 \, \Omega^{a\mu} A^a_\mu + a_4 \, (\partial^\mu \overline{c}^a) A^a_\mu + a_5 \, L^a c^a \\ &+ a_6 \, L^i c^i + a_7 (\partial^\mu A^i_\mu) \omega^i + a_8 \, \lambda c^i \omega^i + a_9 \, \omega^i \vartheta^i + a_{10} \, g f^{abi} \omega^i \overline{c}^a c^b \\ &+ a_{11} \, \lambda J + a_{12} \, \lambda A^a_\mu A^{a\mu} + a_{13} \, \lambda A^i_\mu A^{i\mu} + a_{14} \, \lambda \overline{c}^a c^a + a_{15} \, g f^{abc} \overline{c}^a \overline{c}^b c^c \\ &+ a_{16} \, g f^{abi} \overline{c}^c \overline{c}^b c^i + a_{17} \, \overline{c}^a b^a + a_{18} \, g f^{abi} \overline{c}^a A^b_\mu A^{i\mu} + a_{19} \, g f^{abi} \overline{c}^i \overline{c}^a c^b \\ &+ a_{20} \, b^i \overline{c}^i + a_{21} \, \lambda \overline{c}^i c^i + a_{22} \, \overline{c}^i \vartheta^i + a_{23} \, b^i \omega^i \Big]\end{aligned} \tag{2.17}$$

The identities (2.13) imply that

$$\begin{aligned} &a_1 = a_2 = a_6 = a_7 = a_8 = a_{19} = a_{20} = a_{21} = a_{22} = a_{23} = 0 \,, \\ &a_{10} = -a_5 \,, a_{12} = -\frac{a_4}{2} + \frac{a_5}{2} \,, a_{14} = -2a_{16} + \alpha a_5 \,, a_{15} = \frac{a_{16}}{2} \,, a_{17} = -a_{16} \,, a_{18} = a_4 \,. \end{aligned} \tag{2.18}$$

If we rename the six independent coefficients a_3, a_4, a_5, a_9, a_{11}, a_{16}, according to

$$a_3 \to a_1, \quad a_4 \to -a_3, \quad a_5 \to a_2, \quad a_9 \to \frac{a_5 \chi}{2}, \quad a_{11} \to \frac{a_6 \zeta}{2}, \quad a_{16} \to -\alpha a_4 \,, \tag{2.19}$$

the final expression for $\Delta^{(-1)}$ is found to be

$$\Delta^{(-1)} = \int d^4x \left[a_1\, \Omega^{a\mu} A^a_\mu + a_2 \left(L^a c^a - g f^{abi} \omega^i \overline{c}^a c^b + \frac{1}{2}\lambda A^a_\mu A^{a\mu} + \alpha \lambda \overline{c}^a c^a \right) \right. \tag{2.20}$$
$$+ a_3 \left(\overline{c}^a D^{ab}_\mu A^{b\mu} + \frac{1}{2}\lambda A^a_\mu A^{a\mu} \right) + \alpha a_4 \left(\overline{c}^a b^a - g f^{abi} \overline{c}^a \overline{c}^b c^i - \frac{g}{2} f^{abc} \overline{c}^a \overline{c}^b c^c \right.$$
$$\left. \left. + 2\lambda \overline{c}^a c^a \right) + \frac{a_5 \chi}{2}\, \omega^i \vartheta^i + \frac{a_6 \zeta}{2} \lambda J \right] .$$

At the end, $\Sigma_{\rm CT}$, in (2.16), contains seven free independent parameters a_k $(k = 0, 1, \ldots, 6)$. These parameters can be reabsorbed by means of a multiplicative renormalization of the parameters $\xi = (g, \alpha, \zeta, \chi)$, of the fields $\Phi = (A^{a,i}_\mu, c^{a,i}, \overline{c}^{a,i}, b^{a,i})$ and sources $\phi = (\Omega^{a,i}_\mu, L^{a,i}, \lambda, J, \omega^i, \vartheta^i)$, according to

$$\Sigma(\Phi_0, \phi_0, \xi_0) = \Sigma(\Phi, \phi, \xi) + \eta \Sigma_{\rm CT}(\Phi, \phi, \xi) \, , \tag{2.21}$$

where,

$$\Phi_0^{\rm diag} = Z_\Phi^{1/2} \Phi^{\rm diag} \, , \qquad \Phi_0^{\text{off-diag}} = \widetilde{Z}_\Phi^{1/2} \Phi^{\text{off-diag}} \, , \tag{2.22}$$
$$\phi_0^{\rm diag} = Z_\phi \phi^{\rm diag} \, , \qquad \phi_0^{\text{off-diag}} = \widetilde{Z}_\phi \phi^{\text{off-diag}} \, , \tag{2.23}$$
$$\xi_0 = Z_\xi \xi . \tag{2.24}$$

More precisely, a little algebra results in

$$Z_g = 1 - \eta \frac{a_0}{2} \, , \widetilde{Z}_A = 1 + \eta \left(a_0 + 2a_1 \right) , Z_c = 1 + \eta \left(a_2 + a_3 \right) , \widetilde{Z}_c = 1 - \eta \left(a_2 - a_3 \right) ,$$
$$Z_\alpha = 1 + \eta(a_0 - 2a_3 + 2a_4), Z_\chi = 1 - \eta(a_0 - 2a_2 - 2a_3 - a_5), Z_\zeta = 1 + \eta(2a_0 - 2a_2 - 2a_3 + a_6), \tag{2.25}$$

and

$$Z_A = Z_g^{-2}, Z_b = Z_g^2, \widetilde{Z}_b = Z_g^2 Z_c \widetilde{Z}_c, Z_{\overline{c}} = Z_c^{-1}, \widetilde{Z}_{\overline{c}} = \widetilde{Z}_c, Z_\Omega = Z_c^{-1/2}, \widetilde{Z}_\Omega = Z_g^{-1} \widetilde{Z}_A^{-1/2} Z_c^{-1/2},$$
$$Z_L = Z_g^{-1} Z_c^{-1}, \widetilde{Z}_L = Z_g^{-1} Z_c^{-1/2} \widetilde{Z}_c^{-1/2}, Z_\lambda = Z_g Z_c^{1/2}, Z_J = Z_g^2 Z_c \, Z_\omega = Z_g^{-2} Z_c^{-3/2}, Z_\vartheta = Z_g^{-1} Z_c^{-1} \tag{2.26}$$

Before closing this section, we notice that there is no mixing at all between the mass operator coupled to J and the ghost operator coupled to ϑ^i.

3. Construction of the effective potential

We shall employ dimensional regularization in $d = 4 - \varepsilon$ dimensions. The part of the action (2.1) that we need is obtained by setting all external sources equal to zero, except ϑ^i which is coupled to the operator $g f^{abi} \overline{c}^a c^b$. For the moment, we also discard the mass operator, and concentrate purely on the dimension two ghost operator. For further analysis, we prefer to use the operator $f^{abi} \overline{c}^a c^b$, obtained by a suitable rescaling of the original operator coupled to the source ϑ^i. Therefore, the starting action yields

$$S = S_{\rm YM} + S_{\rm MAG} + S_{\rm diag} + \int d^4x \left(f^{abi} \vartheta^i \overline{c}^a c^b + \frac{\chi}{2} \vartheta^i \vartheta^i \right) \, . \tag{3.1}$$

We define the anomalous dimension $\gamma(g^2)$ of the ghost operator via

$$\overline{\mu}\frac{\partial}{\partial\overline{\mu}}\left[f^{abi}\overline{c}^a c^b\right] = \gamma(g^2)\left[f^{abi}\overline{c}^a c^b\right] = \left(\overline{\mu}\frac{\partial}{\partial\overline{\mu}}\ln Z_\vartheta\right)\left[f^{abi}\overline{c}^a c^b\right] \ . \tag{3.2}$$

From the bare action associated to (3.1), we deduce

$$\frac{1}{2}\chi_o\vartheta^i_o\vartheta^i_o = \frac{1}{2}\overline{\mu}^{-\varepsilon}(\chi+\delta\chi)\vartheta^i\vartheta^i \ . \tag{3.3}$$

The so-called LCO parameter χ is needed to ensure multiplicative renormalizability: a counterterm $\propto \vartheta^2$ is needed to kill the divergences in the Green function $\langle f^{abi}\overline{c}^a(x)c^b(x)f^{abi}\overline{c}^a(y)c^b(y)\rangle$, or equivalently in the generating functional $\mathcal{W}(\vartheta)$. It is clear that divergences $\propto\vartheta^2$ can and do arise. In principle, χ is a free parameter. However, as we do not want to introduce an independent coupling, we shall reexpress χ in terms of the gauge coupling g^2, in such a way that the compatibility with the renormalization group is preserved [19]. We can derive the RG equation for the LCO parameter χ from (3.3),

$$\overline{\mu}\frac{\partial}{\partial\overline{\mu}}\chi = \left(\beta(g^2)\frac{\partial}{\partial g^2}+\gamma_\alpha(g^2)\alpha\frac{\partial}{\partial\alpha}\right)\chi = 2\gamma(g^2)\chi+\delta(g^2) \ , \tag{3.4}$$

where we defined

$$\delta(g^2) = \left(\varepsilon+2\gamma(g^2)-\beta(g^2)\frac{\partial}{\partial g^2}-\alpha\gamma_\alpha(g^2)\frac{\partial}{\partial\alpha}\right)\delta\chi \ . \tag{3.5}$$

Apparently, we require explicit knowledge of $\beta(g^2)$, $\gamma_\alpha(g^2)$ and $\delta(g^2)$ before we can fix $\chi(g^2)$ by solving (3.4). A complete three loop renormalization of QCD in the MAG in arbitrary colour group has already been carried out in [35]. The only missing information is in fact the RG function $\delta(g^2)$ as defined in (3.5) and the anomalous dimension $\gamma(g^2)$ of the ghost operator. To deduce $\delta(g^2)$ we follow the method derived in [36]. There the divergences contributing to the counterterm analogous to $\delta\chi$ were deduced in the massless theory by considering the corresponding ϑ^i 2-point function with no internal ϑ^i propagators. As the Feynman graphs are massless and we are only interested in the divergences, the MINCER [37, 38], algorithm written in the symbolic manipulation language FORM [39], can be used. The Feynman diagrams are generated automatically using the QGRAF package, [40], and for our current problem there are one one loop and twelve two loop Feynman diagrams to determine. In addition we have also carried out the explicit renormalization of the operator $f^{abi}\overline{c}^a c^b$ itself at two loops and verified that the relation derived from the Ward identities, $\gamma(g^2) = -2\gamma_c(g^2)$ holds, suitably adapted to our conventions here. This can be regarded as an extra check on both the algebraic renormalization result as well as the intricate symbolic manipulation required to derive anomalous dimensions in the MAG due to the difficulties arising from the split colour group. See, for instance, [35]. Hence, using the $\overline{\text{MS}}$

renormalization scheme, we obtained the following results for a general gauge group

$$\begin{aligned}
\delta(g^2) &= \delta_0 + \delta_1 g^2 + \delta_2 g^4 + \cdots \,, \\
\delta_0 &= -\frac{C_A}{8\pi^2} \,, \delta_1 = -\frac{1}{2N_A^o\,(16\pi^2)^2}\left(N_A^o C_A^2(\alpha+5) + N_A^d C_A^2(-2\alpha+22)\right) \,, \\
\delta_2 &= -\frac{1}{32(N_A^o)^2\,(16\pi^2)^3}\left((N_A^o)^2(C_A^3(6\alpha^2+78\alpha+402) - 240C_A^2T_FN_f)\right. \\
&\quad + N_A^oN_A^d(C_A^3(60\alpha^2+96\alpha\zeta_3+634\alpha+480\zeta_3+1111) - 608C_A^2T_FN_f) \\
&\quad \left. + (N_A^d)^2(C_A^3(112\alpha^2-192\alpha\zeta_3+276\alpha+2112\zeta_3-1462))\right) \,,
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\gamma(g^2) &= \gamma_0 g^2 + \gamma_1 g^4 + \cdots \,, \\
\gamma_0 &= \frac{1}{2N_A^o\,(16\pi^2)}\left(N_A^o C_A(\alpha+3) + N_A^d C_A(2\alpha+6)\right) \,, \\
\gamma_1 &= \frac{1}{48(N_A^o)^2\,(16\pi^2)^2}\left((N_A^o)^2(C_A^2(6\alpha^2+66\alpha+190) - 80C_AT_FN_f)\right. \\
&\quad \left. + N_A^oN_A^d(C_A^2(54\alpha^2+354\alpha+323) - 160C_AT_FN_f) + (N_A^d)^2C_A^2(60\alpha^2+372\alpha-510)\right) \,.
\end{aligned} \tag{3.7}$$

Using the same notation as [35], N_A is the dimension of the adjoint representation, whereby N_A^d and N_A^o represent the number of diagonal, respectively off-diagonal, generators. Of course, $N_A^d + N_A^o = N_A$. N_f is the number of quark flavours, while T_F and C_A are Casimir operators. Specifying to SU(N), one has $N_A^d = N-1$, $N_A^o = N(N-1)$, $T_F = \frac{1}{2}$ and $C_A = N$.

For simplicity, we shall only determine the potential in the case of SU(2) as gauge group without flavours. If $N = 2$, there is only one ghost condensate, as SU(2) has only one U(1) subgroup. In that case, we have

$$\begin{aligned}
\delta_0 &= -\frac{1}{4\pi^2} \,, & \delta_1 &= -\frac{32}{(16\pi^2)^2} \,, \\
a_0 &= \left(-2\alpha + \frac{8}{3} - \frac{6}{\alpha}\right)\frac{1}{16\pi^2} \,, & a_1 &= \frac{1}{3}\left(-12\alpha^2 - 156\alpha + 52 + \frac{20}{\alpha}\right)\frac{1}{(16\pi^2)^2} \,,
\end{aligned} \tag{3.8}$$

while

$$\begin{aligned}
\beta(g^2) &= -\varepsilon g^2 - 2\left(\beta_0 g^4 + \beta_1 g^6\right) + \cdots \,, \\
\beta_0 &= \frac{22}{3}\frac{1}{16\pi^2} \,, \qquad \beta_1 = \frac{136}{3}\frac{1}{(16\pi^2)^2} \,.
\end{aligned} \tag{3.9}$$

Equation (3.4) can be solved by making χ a Laurent series in g^2,

$$\chi(g^2,\alpha) = \frac{\chi_0(\alpha)}{g^2} + \chi_1(\alpha) + \cdots \,. \tag{3.10}$$

Substituting this in (3.4), we obtain the following differential equations in α for the first two coefficients χ_0 and χ_1.

$$2\beta_0\chi_0 + \alpha a_0\frac{\partial\chi_0}{\partial\alpha} = 2\gamma_0\chi_0 + \delta_0 \,, \tag{3.11}$$

$$2\beta_1\chi_0 + \alpha a_0\frac{\partial\chi_1}{\partial\alpha} + \alpha a_1\frac{\partial\chi_0}{\partial\alpha} = 2\gamma_0\chi_1 + 2\gamma_1\chi_0 + \delta_1 \,. \tag{3.12}$$

Solving yields

$$\chi_0 = \frac{6\alpha + C_0}{3\alpha^2 - 4\alpha + 9} \ . \tag{3.13}$$

For χ_1, we have not been able to find a closed expression. An integral representation is given by

$$\begin{aligned}\chi_1 = & \frac{e^{-\frac{22}{\sqrt{23}}\mathrm{ArcTan}\left(\frac{3\alpha}{\sqrt{23}}-\frac{2}{\sqrt{23}}\right)}}{3\alpha^2-4\alpha+9} \\ & \times\int_{C_1}^{\alpha}\left[\left(\frac{567}{4}x-468x^2+\frac{95}{2}C_0-\frac{129}{2}C_0x+15C_0x^3+54x^3+\frac{153}{4}x^4-\frac{27}{4}x^5+\frac{1107}{4}\right)\right. \\ & \left.\times\frac{e^{\frac{22}{\sqrt{23}}\mathrm{ArcTan}\left(\frac{3x}{\sqrt{23}}-\frac{2}{\sqrt{23}}\right)}}{\pi^2(3x^2-4x+9)^2}\right]dx \ .\end{aligned} \tag{3.14}$$

C_0 and C_1 are constants of integration.

Let us recall that the exact vacuum energy itself will not depend on the choice of the gauge parameter α, which can be proven completely similarly as we already did before in [20, 21]. We also recall that we introduced a method to circumvent the gauge parameter dependence of the explicitly calculated $E_{\rm vac}$, caused by the fact that we are forced to work at a finite order, so that we never obtain that $J = 0$ *exactly*. Essentially, we introduced a "compensating" gauge dependent function that was determined to remove the gauge dependence. If we introduce the following unity

$$1 = \mathcal{N}\int \mathcal{D}\sigma e^{-\frac{i}{2\chi}\int d^4x\left(\frac{\sigma^i}{g}-\chi\theta^i-f^{abi}\overline{c}^a c^b\right)^2} \ , \tag{3.15}$$

with $\mathcal{N}$ the appropriate normalization, we are led to the following action

$$S' = S_{\rm YM} + S_{\rm MAG} + S_{\rm diag} + \int d^4x\left(-\frac{\sigma^i\sigma^i}{2g^2\chi}+\frac{1}{\chi g}\sigma^i f^{abi}\overline{c}^a c^b - \frac{1}{2\chi}f^{abi}f^{cdi}\overline{c}^a c^b\overline{c}^c c^d + \theta^i\frac{\sigma^i}{g}\right) \ , \tag{3.16}$$

with the identification

$$\left\langle g f^{abi}\overline{c}^a c^b\right\rangle = \left\langle \sigma^i\right\rangle \ . \tag{3.17}$$

Following the analysis of [20, 21], one can show that the vacuum energy, given in terms of the effective potential $V(\sigma)$ as

$$E_{\rm vac} = V(\sigma_*) \ , \qquad \text{with } \sigma_* \text{ the global minimum of } V(\sigma) \ , \tag{3.18}$$

shall formally not depend on the gauge parameter, making use of the BRST symmetry which can be extended naturally to the extra field by means of

$$s\sigma^i = s(g f^{abi}\overline{c}^a c^b) = g f^{abi} b^a c^b - g^2 f^{abi}f^{bci}\overline{c}^a c^c c^i - \frac{g^2}{2}f^{abi}f^{bcd}\overline{c}^a c^c c^d \ . \tag{3.19}$$

We shall not repeat the proof here, as it would be merely a notational adaptation of the analogous results in [20, 21]. We emphasize the use of the word *formally*, as we are forced

to work at a finite order. The gauge parameter independence proof is only valid when we would work to all orders. The problem relies on the fact that an important step in the quoted proof is that the sources ϑ^i become zero when the gap equation leading to the minimum of the effective potential is solved. However, as the effective potential $V(\sigma)$ itself shall only be calculated in a loop expansion, we shall only have $\vartheta^i = 0$ up to a certain order, because $\vartheta^i \sim \frac{\partial V(\sigma)}{\partial \sigma^i}$. Consequently, at finite order, residual α-dependence will slip into the final expression for the vacuum energy. To cure the α-dependence at finite order precision, we shall rely on the formalism developed [20, 21]. We apply a transformation to the fields and the sources,

$$\sigma^i = \frac{\widetilde{\sigma}}{\mathcal{F}(g^2,\alpha)} , \qquad \vartheta^i = \widetilde{\vartheta}^i \mathcal{F}(g^2,\alpha) , \tag{3.20}$$

with

$$\mathcal{F}(g^2,\alpha) = 1 + f_0(\alpha) g^2 + f_1(\alpha) g^4 + \cdots , \tag{3.21}$$

to arrive at the following action

$$S' = S_{\rm YM} + S_{\rm MAG} + S_{\rm diag} + \int d^4x \left(-\frac{\widetilde{\sigma}^i \widetilde{\sigma}^i}{2g^2 \mathcal{F}(g^2,\alpha)\chi} + \frac{1}{g\chi \mathcal{F}(g^2,\alpha)} \widetilde{\sigma}^i f^{abi} \overline{c}^a c^b \right.$$
$$\left. - \frac{1}{2\chi} f^{abi} f^{cdi} \overline{c}^a c^b \overline{c}^c c^d + \widetilde{\vartheta}^i \frac{\widetilde{\sigma}^i}{g} \right) . \tag{3.22}$$

In the SU(2) case, in which case there is only one field $\sigma \equiv \sigma^3$, the tree level off-diagonal ghost propagator will read

$$\langle \overline{c}^a c^b \rangle_q = i \frac{-\delta^{ab} q^2 + v \epsilon^{ab}}{q^4 + v^2} , \tag{3.23}$$

where we set

$$v = \frac{g}{\chi_0} \langle \widetilde{\sigma} \rangle . \tag{3.24}$$

We notice that the actions (3.16) and (3.22) are *exactly* equivalent as they are connected via the transformations (3.20), however when working up to a certain order the coefficient functions $f_i(\alpha)$ can enter the results. We shall precisely use these to enforce the gauge parameter independence of the vacuum energy. We shall demand that

$$\frac{dE_{\rm vac}}{d\alpha} = 0 \Rightarrow \text{first order differential equations in } \alpha \text{ for } f_i(\alpha) . \tag{3.25}$$

As an initial condition for the vacuum energy, we shall use the Landau gauge result. In [32], we analyzed the ghost condensate $\langle f^{ABC} \overline{c}^B c^C \rangle$ in the Landau gauge. By connecting the MAG with the Landau gauge in [21], we argued that we can use the Landau gauge as the "initial condition gauge" to match the vacuum energy of any other gauge to that of the Landau gauge, given that the other gauge can be linked to the Landau gauge in a renormalizable fashion. Of course, we should also find a renormalizable interpolating ghost operator. In the present case, it is given by the expression

$$\int d^4x \left(\vartheta^i f^{abi} \overline{c}^a c^b + \kappa \vartheta^c \left(f^{abc} \overline{c}^a c^b + f^{jbc} \overline{c}^j c^b + f^{ajc} \overline{c}^a c^j \right) + \chi \vartheta^i \vartheta^i + \kappa \chi' \vartheta^c \vartheta^c \right) , \tag{3.26}$$

where k is an interpolating parameter[2] and ϑ^i, ϑ^c external sources. The Landau gauge vacuum energy was established to be

$$E_{\rm vac}^{\rm Landau} = -\frac{1}{32\pi^2} e^{\frac{56}{33}} \Lambda_{\overline{\rm MS}}^4 \approx -0.017 \Lambda_{\overline{\rm MS}}^4 \,. \tag{3.27}$$

The SU(2) MAG effective action reads at one loop, again using the $\overline{\rm MS}$ scheme,

$$V_1(\sigma) = \frac{\sigma^2}{2\chi_0}\left(1 - \frac{\chi_1}{\chi_0} g^2\right) + \frac{1}{32\pi^2}\frac{g^2\sigma^2}{\chi_0^2}\left(\ln\frac{g^2\sigma^2}{\chi_0^2\overline{\mu}^4} - 3\right) \,. \tag{3.28}$$

Performing the transformation yields the potential

$$V_1(\widetilde{\sigma}) = \frac{\widetilde{\sigma}^2}{2\chi_0}\left(1 - \left(\frac{\chi_1}{\chi_0} + 2f_0\right) g^2\right) + \frac{1}{32\pi^2}\frac{g^2\widetilde{\sigma}^2}{\chi_0^2}\left(\ln\frac{g^2\widetilde{\sigma}^2}{\chi_0^2\overline{\mu}^4} - 3\right) \,. \tag{3.29}$$

The gap equation $\frac{dV_1}{d\widetilde{\sigma}} = 0$ leads to

$$\frac{\widetilde{\sigma}}{\chi_0}\left(1 - \left(\frac{\chi_1}{\chi_0} + 2f_0\right) g^2\right) + \frac{1}{16\pi^2}\frac{g^2\widetilde{\sigma}}{\chi_0^2}\left(\ln\frac{g^2\widetilde{\sigma}^2}{\chi_0^2\overline{\mu}^4} - 3\right) + \frac{1}{16\pi^2}\frac{g^2\widetilde{\sigma}}{\chi_0^2} = 0 \,. \tag{3.30}$$

Assuming that v_* is a solution of the previous equation written in terms of the variable v as defined in (3.24), we obtain as vacuum energy

$$E_{\rm vac}^{\rm MAG} = -\frac{1}{32\pi^2} v_*^2 \,. \tag{3.31}$$

Now, by construction of the method, the functions $f_i(\alpha)$ are fixed to ensure that $E_{\rm vac}^{\rm Landau} = E_{\rm vac}^{\rm MAG}$. Doing so, we can in fact solve

$$-\frac{1}{32\pi^2} v_*^2 = -\frac{1}{32\pi^2} e^{\frac{56}{33}} \Lambda_{\overline{\rm MS}}^4 \,, \tag{3.32}$$

or

$$v_* = e^{\frac{28}{33}} \Lambda_{\overline{\rm MS}}^2 \approx 2.34 \Lambda_{\overline{\rm MS}}^2 \,. \tag{3.33}$$

For comparison, the lattice group of [34] quote a (preliminary) estimate of $v \approx 1.3\,{\rm GeV}^2$. Using $\Lambda_{\overline{\rm MS}} \approx 275\,{\rm MeV}$ in the case of SU(2) [41], we find

$$v_* \approx 0.18\,{\rm GeV}^2 \,. \tag{3.34}$$

It is instructive to have a look at the effective coupling constant. Assuming that we solve the gap equation at a scale $\overline{\mu}^2 = v_*^2$ in order to kill large logarithms and using the one loop result (3.30), we deduce that, for any α,

$$\left.\frac{g^2 N}{16\pi^2}\right|_{N=2} = \frac{9}{28} \,, \tag{3.35}$$

which is sufficiently small to speak about at least qualitatively acceptable results. We notice that our value is considerable smaller than the lattice value. However, continuum

[2]We recall that α, κ are gauge parameters, such that the MAG corresponds to $\kappa = 0$ while the Landau gauge to $(\alpha = 0, \kappa = 1)$ [21].

effects should still be investigated on the lattice, while we employed perturbation theory at lowest order. One can imagine other sources of nonperturbative effects that contributes to the condensate. Anyhow, analytical continuum calculations as well as lattice simulations seem to favour a nonvanishing ghost condensate.

We also see that we do not need explicit knowledge of the $f_i(\alpha)$-terms to obtain the desired results, by taking into account how the functions $f_i(\alpha)$ are fixed. For completeness, one could determine $f_0(\alpha)$ by matching the solution of the gap equation (3.30) at $\overline{\mu}^2 = v_*$, being

$$\left.\frac{g^2 N}{16\pi^2}\right|_{N=2} = \frac{1 - \frac{1}{8\pi^2\chi_0}}{8\pi^2\left(\frac{\chi_1}{\chi_0} + 2f_0\right)} , \tag{3.36}$$

with the already fixed solution (3.35), so there is absolutely no need to solve the defining differential equations (3.25).

4. Consequences of the ghost condensate

First of all, there is the obvious breaking of the δ-symmetry (1.12) since $\langle f^{abi}\overline{c}^a c^b\rangle = \frac{1}{2}\langle\delta(f^{abi}\overline{c}^a\overline{c}^b)\rangle$. Let us have a look at the associated current in the SU(2) case under study,[3] while setting $\alpha = 0$ in (1.1) to immediately recover the MAG (1.11). Doing so, the symmetry generator δ, defined in (1.12), is given by[4]

$$\delta = \int d^4x \left(c^a \frac{\delta}{\delta \overline{c}^a} + g\epsilon^{ab} c^b c \frac{\delta}{\delta b^a} \right) \tag{4.1}$$

in the functional form, so that we find after its application on the action (1.1),

$$0 = \delta S = \int d^4x \delta\mathcal{L} \Rightarrow \left(c^a \frac{\delta S}{\delta \overline{c}^a} + g\epsilon^{ab} c^b c \frac{\delta S}{\delta b^a} \right) = \partial_\mu (c^a D^{ab}_\mu c^b) . \tag{4.2}$$

After using the equations of motion, it follows that

$$\mathcal{K}_\mu = c^a D^{ab}_\mu c^b \tag{4.3}$$

is the associated conserved current, $\partial_\mu \mathcal{K}_\mu = 0$. Now, it turns out that this current $\mathcal{K}_\mu$ can be brought into the following useful form

$$\mathcal{K}_\mu = s(A^a_\mu c^a) , \tag{4.4}$$

which can be checked by using the definition of the BRST transformation s, as given in (1.7).

We would like to point out here that the ghost condensate $\left\langle \epsilon^{ab}\overline{c}^a c^b \right\rangle$ does not break the BRST symmetry, since $s(\epsilon^{ab}\overline{c}^a c^b) \neq 0$, so that due to the nilpotency, we certainly do have $\epsilon^{ab}\overline{c}^a c^b \neq s(\ldots)$, meaning that the ghost condensate is not an order parameter for the BRST symmetry. In order to avoid confusion, let us mention that, in the case that

[3] A little more cumbersome calculations will lead to the same conclusion in the general SU(N) case.

[4] $\epsilon^{12} = -\epsilon^{21} = 1$, $\epsilon^{11} = \epsilon^{22} = 0$.

one would study the condensates $\langle \epsilon^{ab} \overline{c}^a \overline{c}^b \rangle$ and $\langle \epsilon^{ab} c^a c^b \rangle$, corresponding to an equivalent vacuum, then a nilpotent BRST charge also exists. We refer to [32] for a detailed discussion of the completely analogous arguments in the Landau gauge.

As we have already mentioned, there exists another version of the MAG [29], in which case the diagonal gauge fixing also respects the anti-BRST symmetry $\overline{s}$. The ghost condensate discussed here then breaks this anti-BRST symmetry. We did choose the diagonal gauge fixing (1.4), as this corresponds to the Landau gauge for the diagonal sector, a gauge also used in the lattice simulations corresponding to the MAG [24, 25, 34].

Returning to the current $\mathcal{K}_\mu$ written down in (4.4), we can use the fact that it is BRST exact. As a consequence, the Goldstone boson associated with the broken δ-invariance will cancel from the physical spectrum, as it will belong to a BRST exact state, and physical states are defined as BRST invariant states, modulo the (trivially) invariant exact states. We rely here on the fact that the current corresponding to a spontaneously broken symmetry stands in a direct correspondence with the associated Goldstone boson [1], namely the current can be used to create/annihilate the Goldstone boson. Inserting this current into a physical gauge invariant, and thus a fortiori BRST invariant, correlator then immediately leads to a vanishing result.

A nonvanishing ghost condensate also strongly influences the behaviour of the off-diagonal ghost propagator, (3.23). We notice the safe infrared behaviour when $p^2 \to 0$. However, the ghost condensate will also enter the off-diagonal gluon propagator through radiative corrections. At one loop order, the quartic $A^2 \overline{c} c$ coupling will give rise to an effective 1PI off-diagonal gluon mass δM^2 given by [18]

$$\delta M^2 = -\frac{g^2}{16\pi^2} v_* < 0 \, . \tag{4.5}$$

Clearly, this would be a tachyonic off-diagonal gluon mass, indicative of instabilities in the ghost condensed vacuum. There is however a resolution to this problem.

So far we have only considered the contribution of the ghost condensate to an effective gluon mass. If we assume that we would have started with a sufficiently large positive tree level off-diagonal gluon mass squared, the *loop effect* (4.5) should merely introduce a shift in the tree level value, together with potential other shifts coming from the other interactions.

We have investigated the dynamical generation of a off-diagonal gluon mass m_{od}^2 with similar techniques as employed here [21]. We considered the dimension 2 operator $\frac{1}{2} A_\mu^a A_\mu^a + \alpha \overline{c}^a c^a$, which is on-shell BRST invariant as encoded in the integrated λ-equation (2.11), and successfully constructed the effective potential at one loop in the SU(2) case, leading to a finite value

$$m_{od}^2 = \sqrt{\frac{3}{2} e^{\frac{17}{6}}} \Lambda_{\overline{\rm MS}}^2 \approx 5.05 \Lambda_{\overline{\rm MS}}^2 \, , \tag{4.6}$$

in the MAG limit $\alpha \to 0$. In a meaningful perturbative expansion, one should certainly have $\frac{g^2 N}{16\pi^2} < 1$, so that upon comparing the numbers (3.33), (3.35), (4.5) and (4.6), we conclude that the ghost condensation will induce a negative shift in the mass (4.6), however the net result will be still positive. Further one loop corrections will come from the pure gluonic vacuum polarization. A complete account of similar effects in the Landau gauge was presented in [26].

Let us finally mention that the diagonal sector will not be influenced by the ghost nor gluon condensate, since the $\mathrm{U}(1)^{N-1}$ Ward identity (2.10) forbids a diagonal gluon mass, while the diagonal antighost equation (2.9) excludes a diagonal ghost mass.

5. Conclusion

We have given evidence for the existence of a mass dimension 2 ghost condensate in the MAG. We used the LCO formalism [19] to construct a sensible effective potential for the ghost operator $f^{abi}\overline{c}^a c^b$. We proved the renormalizability to all orders of perturbation theory, and explicitly calculated the one loop effective potential, thereby finding a non-vanishing ghost condensate since $\langle f^{abi}\overline{c}^a c^b\rangle$ corresponds to a vacuum with lower energy. The mere existence of a ghost condensate is in qualitative agreement with recent lattice data [33, 34].

There are a few interesting open questions related to the ghost condensate that deserves further investigation. Since $\langle f^{abi}\overline{c}^a c^b\rangle$ serves as an order parameter for a symmetry, it should be investigated whether this symmetry might get restored if we would allow for finite temperature effects.

In [42, 43], it was discussed how a (partial) treatment of Gribov copy effects in the MAG might be handled via a restriction of the domain of path integration along the lines of Gribov's original approach [4]. Since this restriction also seriously alters the infrared behaviour of the propagators, it would be instructive to find out whether there is a significant change in the obtained values of the ghost condensate.

So far, the mass generating mechanism in the Landau gauge or MAG was in fact depending on the gauge, since the used operator is not gauge invariant, nevertheless the qualitative features of the analytical results in the MAG [21] are in quite good agreement with lattice data [24, 25]. In principle, A^2 is gauge invariant when used in the Landau gauge, as it is formally equivalent to the gauge invariant functional A^2_{min}, obtained by taking the absolute minimum of A^2 along its gauge orbit. However, outside of the Landau gauge, it is unclear how to use this operator. In [44 – 46], we developed a local, renormalizable non-Abelian gauge invariant action, based on the nonlocal mass operator $F\frac{1}{D^2}F$, which could serve as a starting point to discuss a gauge invariant mechanism behind dynamical mass parameters in e.g. the gluon propagator (or physical correlators). Due to the gauge invariance, we therefore expect the *same* tree level mass in the diagonal and off-diagonal sector, even in the MAG. The question arises what might cause the possible difference between the diagonal and off-diagonal sector? A possible explanation might be the ghost condensate, in a fashion similar to what we studied in [26]. Said otherwise, the ghost condensate could play an important role clarifying the mechanism(s) behind Abelian dominance.

Acknowledgments

The Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq-Brazil), the SR2-UERJ and the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior

(CAPES) are gratefully acknowledged for financial support. D. Dudal is supported by the "Special Research Fund" of Ghent University.

References.

[1] M.E. Peskin and D.V. Schroeder, *An introduction to quantum field theory*, Addison-Wesley Publishing Company (1995).

[2] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, Princeton University Press (1992).

[3] O. Piguet and S.P. Sorella, *Algebraic renormalization: perturbative renormalization, symmetries and anomalies*, *Lect. Notes Phys.* **M28** (1995) 1.

[4] V.N. Gribov, *Quantization of non-abelian gauge theories*, *Nucl. Phys.* **B 139** (1978) 1.

[5] J.A. Gracey, *One loop gluon form factor and freezing of α_s in the Gribov-Zwanziger QCD Lagrangian*, *JHEP* **05** (2006) 052 [hep-ph/0605077].

[6] D. Dudal, R.F. Sobreiro, S.P. Sorella and H. Verschelde, *The Gribov parameter and the dimension two gluon condensate in euclidean Yang-Mills theories in the Landau gauge*, *Phys. Rev.* **D 72** (2005) 014016 [hep-th/0502183].

[7] T. Kugo and I. Ojima, *Local covariant operator formalism of nonabelian gauge theories and quark confinement problem*, *Prog. Theor. Phys. Suppl.* **66** (1979) 1 [*Erratum Prog. Theor. Phys.* **71** (1984) 1121].

[8] T. Kugo, *The universal renormalization factors Z_1/Z_3 and color confinement condition in non-abelian gauge theory*, hep-th/9511033.

[9] A. Maas, A. Cucchieri and T. Mendes, *On the infrared behavior of Green's functions in Yang-Mills theory*, *Braz. J. Phys.* **37** (2007) 219 [hep-lat/0610006].

[10] P. Watson and R. Alkofer, *Verifying the Kugo-Ojima confinement criterion in Landau gauge QCD*, *Phys. Rev. Lett.* **86** (2001) 5239 [hep-ph/0102332].

[11] C. Lerche and L. von Smekal, *On the infrared exponent for gluon and ghost propagation in Landau gauge QCD*, *Phys. Rev.* **D 65** (2002) 125006 [hep-ph/0202194].

[12] D. Zwanziger, *Non-perturbative Landau gauge and infrared critical exponents in QCD*, *Phys. Rev.* **D 65** (2002) 094039 [hep-th/0109224].

[13] M. Schaden, *Mass generation in continuum* SU(2) *gauge theory in covariant abelian gauges*, hep-th/9909011.

[14] K.-I. Kondo and T. Shinohara, *Abelian dominance in low-energy gluodynamics due to dynamical mass generation*, *Phys. Lett.* **B 491** (2000) 263 [hep-th/0004158].

[15] G. 't Hooft, *Topology of the gauge condition and new confinement phases in nonabelian gauge theories*, *Nucl. Phys.* **B 190** (1981) 455.

[16] H. Min, T. Lee and P.Y. Pac, *Renormalization of Yang-Mills theory in the abelian gauge*, *Phys. Rev.* **D 32** (1985) 440.

[17] A.R. Fazio, V.E.R. Lemes, M.S. Sarandy and S.P. Sorella, *The diagonal ghost equation ward identity for Yang-Mills theories in the maximal abelian gauge*, *Phys. Rev.* **D 64** (2001) 085003 [hep-th/0105060].

[18] D. Dudal and H. Verschelde, *On ghost condensation, mass generation and abelian dominance in the maximal abelian gauge*, *J. Phys.* **A 36** (2003) 8507 [hep-th/0209025].

[19] H. Verschelde, *Perturbative calculation of nonperturbative effects in quantum field theory*, *Phys. Lett.* **B 351** (1995) 242.

[20] D. Dudal et al., *Dynamical gluon mass generation from $\langle A_\mu^2 \rangle$ in linear covariant gauges*, *JHEP* **01** (2004) 044 [hep-th/0311194].

[21] D. Dudal et al., *An analytic study of the off-diagonal mass generation for Yang-Mills theories in the maximal abelian gauge*, *Phys. Rev.* **D 70** (2004) 114038 [hep-th/0406132].

[22] Z.F. Ezawa and A. Iwazaki, *Abelian dominance and quark confinement in Yang-Mills theories*, *Phys. Rev.* **D 25** (1982) 2681.

[23] K.-I. Kondo, *Vacuum condensate of mass dimension* 2 *as the origin of mass gap and quark confinement*, *Phys. Lett.* **B 514** (2001) 335 [hep-th/0105299].

[24] K. Amemiya and H. Suganuma, *Effective mass generation of off-diagonal gluons as the origin of infrared abelian dominance in the maximally abelian gauge in QCD*, *Phys. Rev.* **D 60** (1999) 114509 [hep-lat/9811035].

[25] V.G. Bornyakov, M.N. Chernodub, F.V. Gubarev, S.M. Morozov and M.I. Polikarpov, *Abelian dominance and gluon propagators in the maximally abelian gauge of* SU(2) *lattice gauge theory*, *Phys. Lett.* **B 559** (2003) 214 [hep-lat/0302002].

[26] M.A.L. Capri et al., *A combined study of the gluon and ghost condensates $\langle A_\mu^2 \rangle$ and $\langle \epsilon^{abc} \bar{c}^b c^c \rangle$ in euclidean* SU(2) *Yang-Mills theory in the Landau gauge*, *Phys. Rev.* **D 73** (2006) 014001 [hep-th/0508216].

[27] T. Suzuki, K. Ishiguro, Y. Mori and T. Sekido, *The dual Meissner effect and abelian magnetic displacement currents*, *Phys. Rev. Lett.* **94** (2005) 132001 [hep-lat/0410001].

[28] G. Curci and R. Ferrari, *On a class of Lagrangian models for massive and massless Yang-Mills fields*, *Nuovo Cim.* **A32** (1976) 151.

[29] D. Dudal, V.E.R. Lemes, M.S. Sarandy, S.P. Sorella and M. Picariello, *On the* $\mathrm{SL}(2,\mathbb{R})$ *symmetry in Yang-Mills theories in the Landau, Curci-Ferrari and maximal abelian gauge*, *JHEP* **12** (2002) 008 [hep-th/0211007].

[30] V.E.R. Lemes, M.S. Sarandy, S.P. Sorella, M. Picariello and A.R. Fazio, *Ghost condensates in Yang-Mills theories in nonlinear gauges*, *Mod. Phys. Lett.* **A 18** (2003) 711 [hep-th/0210036].

[31] V.E.R. Lemes, M.S. Sarandy and S.P. Sorella, *Ghost condensates in Yang-Mills theories in the Landau gauge*, *Ann. Phys. (NY)* **308** (2003) 1 [hep-th/0210077].

[32] D. Dudal et al., *More on ghost condensation in Yang-Mills theory: BCS versus Overhauser effect and the breakdown of the Nakanishi-Ojima annex* $\mathrm{SL}(2,\mathbb{R})$ *symmetry*, *JHEP* **06** (2003) 003 [hep-th/0305020].

[33] A. Cucchieri, T. Mendes and A. Mihara, *Ghost condensation on the lattice*, *Phys. Rev.* **D 72** (2005) 094505 [hep-lat/0508028].

[34] T. Mendes, A. Cucchieri and A. Mihara, *Infrared maximally abelian gauge*, *AIP Conf. Proc.* **892** (2007) 203 [hep-lat/0611002].

JHEP01(2008)006

[35] J.A. Gracey, *Three loop $\overline{MS}$ renormalization of QCD in the maximal abelian gauge*, *JHEP* **04** (2005) 012 [hep-th/0504051].

[36] R.E. Browne and J.A. Gracey, *Two loop effective potential for $\langle A_\mu^2 \rangle$ in the Landau gauge in quantum chromodynamics*, *JHEP* **11** (2003) 029 [hep-th/0306200].

[37] S.G. Gorishnii, S.A. Larin, L.R. Surguladze and F.V. Tkachov, *MINCER: program for multiloop calculations in quantum field theory for the Schoonschip system*, *Comput. Phys. Commun.* **55** (1989) 381.

[38] S.A. Larin, F.V. Tkachov and J.A.M. Vermaseren, *The form version of MINCER*, NIKHEF-H-91-18.

[39] J.A.M. Vermaseren, *New features of FORM*, math-ph/0010025.

[40] P. Nogueira, *Automatic Feynman graph generation*, *J. Comput. Phys.* **105** (1993) 279.

[41] C. Michael, *The running coupling from lattice gauge theory*, *Phys. Lett.* **B 283** (1992) 103 [hep-lat/9205010].

[42] M.A.L. Capri, V.E.R. Lemes, R.F. Sobreiro, S.P. Sorella and R. Thibes, *The influence of the Gribov copies on the gluon and ghost propagators in euclidean Yang-Mills theory in the maximal abelian gauge*, *Phys. Rev.* **D 72** (2005) 085021 [hep-th/0507052].

[43] M.A.L. Capri, V.E.R. Lemes, R.F. Sobreiro, S.P. Sorella and R. Thibes, *A study of the maximal abelian gauge in* SU(2) *euclidean Yang-Mills theory in the presence of the Gribov horizon*, *Phys. Rev.* **D 74** (2006) 105007 [hep-th/0609212].

[44] M.A.L. Capri et al., *A study of the gauge invariant, nonlocal mass operator $Tr \int d^4x F_{\mu\nu}(D^2)^{-1}F_{\mu\nu}$ in Yang-Mills theories*, *Phys. Rev.* **D 72** (2005) 105016 [hep-th/0510240].

[45] M.A.L. Capri et al., *Quantum properties of a non-abelian gauge invariant action with a mass parameter*, *Phys. Rev.* **D 74** (2006) 045008 [hep-th/0605288].

[46] D. Dudal, N. Vandersickel and H. Verschelde, *Unitarity analysis of a non-abelian gauge invariant action with a mass*, *Phys. Rev.* **D 76** (2007) 025006 [arXiv:0705.0871].

JHEP01(2008)006